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MV-Algebra Pasting

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Z. Riečanová proved that every D-lattice is a set-theoretical union of MV-algebras. These MV-algebras are blocks in the D-lattice. There is a dual question: How can we construct a D-poset from a given collection of MV-algebras. To solve this problem we use the "pasting" technique. We define an admissible system of MV-algebras and we prove that the pasting of this system is a D-poset.

KEY WORDS: D-poset; D-lattice; MV-algebra; MV-algebras pasting; block.

1. INTRODUCTION

In 1992, in the study of axiomatic systems of fuzzy sets, Kôpka (1992) defined a new algebraic structure, a so-called *difference poset* (in short a *D-poset*) *of fuzzy sets*, where a difference of comparable fuzzy sets was a primary operation. A generalization of a D-poset of fuzzy sets to an abstract partially ordered set, where a primary operation is a partially defined difference, yields a very general and, at the same time, a very simple structure—a *difference poset* (*a D-poset*). An alternative structure to a D-poset based on a partial binary sum operation is an *effect algebra* (Foulis and Bennett, 1994) (or *unsharp orthoalgebra* (Giuntini and Greuling, 1989)). Although these frameworks are algebraically equivalent, they originated in completely different starting points and they have their original systems of axioms. The similar situation can be seen in the theory of infinite-valued (Lukasiewicz) logics, where Wajsberg algebras (Font *et al.*, 1984) and MV-algebras (Chang, 1957) are the same structures. In the difference posets theory, an MV-algebra is characterized as a D-lattice (lattice ordered D-poset) of pairwise compatible elements. The results of a special direction in MV-algebras and D-posets research can be found in Dvurečenskij and Pulmannová (2000).

Riečanová (2000) proved that every D-lattice is a set-theoretical union of maximal mutually compatible sub-D-lattices (i.e., maximal sub-MV-algebras),

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called blocks. Jenča (2001) generalized this assertion for homogeneous effect algebras.

Now, a natural dual question arises: How can we construct a D-poset from a given collection of MV-algebras? To solve this problem we use the "pasting" technique. A method of construction of quantum logics making use of the pasting of Boolean algebras was originally suggested by Greechie (1971). In the present paper, we give a generalization of this method.

2. BASIC DEFINITIONS AND FACTS

Let $\mathcal{P}$ be a bounded partially ordered set with the least element $0_{\mathcal{P}}$ and the greatest one $1_{\mathcal{P}}$. Let $\ominus$ be a partial binary difference operation on $\mathcal{P}$ such that there exists $b \ominus a$ in $\mathcal{P}$ if and only if $a \leq b$ and the following axioms hold.

(D1) $a \ominus 0_{\mathcal{P}} = a$ for any $a \in \mathcal{P}$.
(D2) $a \leq b \leq c$ implies $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The structure $(\mathcal{P}, \leq, \ominus, 0_{\mathcal{P}}, 1_{\mathcal{P}})$ is called *a difference poset* (*a D-poset*). For the simplicity of the notation, we shall write $\mathcal{P}$ instead of $(\mathcal{P}, \leq, \ominus, 0_{\mathcal{P}}, 1_{\mathcal{P}})$. A lattice-ordered D-poset is called a *D-lattice*.

For any element a in a D-poset, the element $1_{\mathcal{P}} \ominus a$ is called the *orthosupplement* of a and is denoted by $a^{\perp}$. The unary operation $\perp: a \mapsto a^{\perp}$ is an involution ($(a^{\perp})^{\perp} = a$) and order reversing ($a \leq b$ implies $b^{\perp} \leq a^{\perp}$).

A *sum of orthogonal elements*, denoted by $\oplus$, is a dual partial binary operation to a difference defined by the formula

$$a \oplus b := (a^{\perp} \ominus b)^{\perp} \quad \text{for} \quad a, b \in \mathcal{P}, b \leq a^{\perp}.$$

Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in a D-poset $\mathcal{P}$. We define

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n,$$

for any $n \geq 3$, supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in $\mathcal{P}$. We say that a finite system $F = \{a_1, a_2, \ldots, a_n\}$ of a D-poset $\mathcal{P}$ is $\oplus-$*orthogonal*, if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists in $\mathcal{P}$ and then we write

$$a_1 \oplus a_2 \oplus \cdots \oplus a_n = \bigoplus_{i=1}^{n} a_i .$$

An arbitrary system G of $\mathcal{P}$ is $\oplus-$*orthogonal*, if every finite subsystem of G is $\oplus-$orthogonal.

Elements a and b from a D-poset $\mathcal{P}$ are *compatible* ($a \leftrightarrow b$), if there exist $c, d \in \mathcal{P}$ such that $c \leq a \leq d$, $c \leq b \leq d$ and $d \ominus a = b \ominus c$.

If $\mathcal{P}$ is a D-lattice, then $a \leftrightarrow b$ if and only if $(a \vee b) \ominus a = b \ominus (a \wedge b)$.

Kôpka (1995) studied the compatibility in D-posets and defined a *Boolean D-poset*.

A poset $\mathcal{P}$ with the least element $0_{\mathcal{P}}$ and the greatest element $1_{\mathcal{P}}$ is said to be a *Boolean D-poset* if there exists a binary operation "$-$" on $\mathcal{P}$ satisfying the following conditions.

(BD1) $a - 0_{\mathcal{P}} = a$ for any $a \in \mathcal{P}$.
(BD2) $a - (a - b) = b - (b - a)$ for every $a, b \in \mathcal{P}$.
(BD3) $a, b \in \mathcal{P}, a \leq b$ implies $c - b \leq c - a$ for any $c \in \mathcal{P}$.
(BD4) $(a - b) - c = (a - c) - b$ for every $a, b, c \in \mathcal{P}$.

Properties of Boolean D-posets were studied in Chovanec and Kôpka (1997). It was shown that a Boolean D-poset is a D-lattice of pairwise compatible elements and vice versa. (We note that an orthomodular lattice of pairwise compatible elements is a Boolean algebra.)

An *MV-algebra* is an algebra $(\mathcal{A}, +, *, 0, 1)$, where $\mathcal{A}$ is a nonempty set, 0 and 1 are constant elements of $\mathcal{A}$, $+$ is a binary operation and $*$ is a unary operation satisfying the following axioms.

(MVA1) $(a + b) = (b + a)$.
(MVA2) $(a + b) + c = a + (b + c)$.
(MVA3) $a + 0 = a$.
(MVA4) $a + 1 = 1$.
(MVA5) $(a^*)^* = a$.
(MVA6) $0^* = 1$.
(MVA7) $a + a^* = 1$.
(MVA8) $(a^* + b)^* + b = (a + b^*)^* + a$.

The lattice operations $\vee$ and $\wedge$ are defined in any MV-algebra by

$$a \vee b = (a^* + b)^* + b \quad \text{and} \quad a \wedge b = ((a + b^*)^* + b^*)^*.$$

We write $a \leq b$, if $a \vee b = b$. The relation $\leq$ is a partial ordering on $\mathcal{A}$ and $0 \leq a \leq 1$ for any $a \in \mathcal{A}$. An MV-algebra is a distributive lattice with respect to the operations $\vee$ and $\wedge$.

We put

$$a - b = (a^* + b)^* \quad \text{for every} \quad a, b \in A.$$

Then an MV-algebra $\mathcal{A}$ becomes a Boolean D-poset.

Conversely, let $(\mathcal{P}, -, 0_{\mathcal{P}}, 1_{\mathcal{P}})$ be a Boolean D-poset. Let us put

$$a^* = 1_{\mathcal{P}} - a \quad \text{for any} \quad a \in \mathcal{P},$$

and

$$a + b = (a^* - b)^* \quad \text{for every} \quad a, b \in \mathcal{P}.$$

Then $(\mathcal{P}, +, *, 0_{\mathcal{P}}, 1_{\mathcal{P}})$ becomes an MV-algebra. Consequently, Boolean D-posets are algebraically equivalent to MV-algebras. In the present paper, we shall use the notion of an MV-algebra instead of a Boolean D-poset.

By a σ*-complete* D-poset we mean a D-poset $\mathcal{P}$ such that for any countable sequence $\{a_n\}_{n=1}^{\infty}$ of elements of $\mathcal{P}$ the least upper bound $\bigvee_{n=1}^{\infty} a_n$ and the greatest lower bound $\bigwedge_{n=1}^{\infty} a_n$ exist in $\mathcal{P}$.

For reader's convenience we present some properties of a σ-complete MV-algebra $\mathcal{A}$. If $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$, then the following assertions are true for every $b \in \mathcal{A}$.

(i) $b - \bigwedge_{n=1}^{\infty} a_n = \bigvee_{n=1}^{\infty}(b - a_n)$.
(ii) $b - \bigvee_{n=1}^{\infty} a_n = \bigwedge_{n=1}^{\infty}(b - a_n)$.
(iii) $(\bigvee_{n=1}^{\infty} a_n) - b = \bigvee_{n=1}^{\infty}(a_n - b)$.
(iv) $b \wedge (\bigvee_{n=1}^{\infty} a_n) = \bigvee_{n=1}^{\infty}(b \wedge a_n)$.
(v) If $a_i \wedge a_j = 0_{\mathcal{A}}$ for $i \neq j$, then the sequence $\{a_n\}_{n=1}^{\infty}$ is $\oplus$–orthogonal and $\bigoplus_{n=1}^{\infty} a_i = \bigvee_{n=1}^{\infty} a_n$.

A nonzero element a from a D-poset $\mathcal{P}$ is called *an atom* if the inequality $b \leq a$ entails either $b = 0_{\mathcal{P}}$ or $b = a$. A D-poset $\mathcal{P}$ is said to be *atomic* if for any nonzero element $b \in \mathcal{P}$ there exists an atom $a \in \mathcal{P}$ such that $a \leq b$.

3. MV-ALGEBRAS PASTING

Definition 1. Let $\mathcal{P}$ be a D-poset and $\mathcal{N}$ be a set of all nonnegative integers.

(1) An *orthogonal multiple* of an element $a \in \mathcal{P}$ is defined recurrently as follows.
 (i) $0a := 0_{\mathcal{P}}$.
 (ii) $1a := a$.
 (iii) $na := (n-1)a \oplus a$ whenever $(n-1)a \leq a^{\perp}$, $n \geq 2$.
(2) The maximal nonnegative integer $n \in \mathcal{N}$ such that an element na exists in $\mathcal{P}$ is called an *isotropic index* of a and we denote it $\tau(a)$. If na exists for every integer n, then $\tau(a) = \infty$.

It is obvious that the equality $\tau(0_{\mathcal{P}}) = \infty$ holds in every D-poset.

Lemma 2. *A σ-complete D-poset has no nonzero elements with infinite isotropic index.*

Proof: Let $\mathcal{P}$ be a σ-complete D-poset and let for some $a > 0_{\mathcal{P}}$ be $\tau(a) = +\infty$. Denoting by $a_1 = a, a_2 = 2a, \ldots, a_n = na, \ldots$, we put $b = \bigvee_{n=1}^{\infty} a_n$. Then $a \leq b$

and

$$b \ominus a = \left(\bigvee_{n=1}^{\infty} a_n \right) \ominus a = \bigvee_{n=1}^{\infty} (a_n \ominus a)$$

$$= 0_{\mathcal{P}} \vee a \vee 2a \vee \cdots \vee a_{n-1} \vee a_n \vee \cdots = b.$$

This is equivalent to $a = 0_{\mathcal{P}}$, which contradicts the assumption $a > 0_{\mathcal{P}}$. □

We note that an MV-algebra is a Boolean algebra if and only if the isotropic index of every nonzero element is equal to one.

Let $\mathcal{A}$ be an atomic MV-algebra. We shall denote by $\langle \mathcal{A} \rangle$ the set of all atoms of $\mathcal{A}$ and by $|A|$ the cardinality of a set A, where $A \subset \langle \mathcal{A} \rangle$.

Definition 3. Let $\mathcal{S} = \{\mathcal{A}_t : t \in T, T$ is an index set$\}$ be a system of atomic σ-complete MV-algebras. Let A and B be finite sets of atoms such that $A \subset \langle \mathcal{A}_t \rangle$, $B \subset \langle \mathcal{A}_s \rangle$ for $t \neq s$, and $|A| = |B|$. We say that the sets A and B are *equivalent with respect to isotropic indices*, and write $A \sim_\tau B$, if one of the following conditions hold.

(1) $A = \emptyset$ and $B = \emptyset$.
(2) If $a \in A$, then there exists $b \in B$ such that $\tau(a) = \tau(b)$, and moreover, if $a_1, a_2 \in A$ and $a_1 \neq a_2$, then there exist atoms $b_1, b_2 \in B$ such that $\tau(a_1) = \tau(b_1)$, $\tau(a_2) = \tau(b_2)$ and $b_1 \neq b_2$.

We remark that if $A \sim_\tau B$, then there exists a bijection φ from A onto B defined as follows: $\varphi(a) = b$ if and only if $\tau(a) = \tau(b)$.

It is easily seen that $A \sim_\tau B$ implies $B \sim_\tau A$ and, in addition, if $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are three mutually different atomic σ-complete MV-algebras from $\mathcal{S}$ such that $A \subset \langle \mathcal{A} \rangle$, $B \subset \langle \mathcal{B} \rangle$, $C \subset \langle \mathcal{C} \rangle$, $A \sim_\tau B$ and $B \sim_\tau C$, then $A \sim_\tau C$.

Definition 4. Let $\mathcal{S} = \{\mathcal{A}_t : t \in T\}$ be a system of atomic σ-complete MV-algebras and let the following conditions hold for arbitrary three MV-algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ from $\mathcal{S}$.

(1) If $A \subset \langle \mathcal{A} \rangle$, $B \subset \langle \mathcal{B} \rangle$ and $A \sim_\tau B$, then $\mathcal{A} \setminus A \neq \emptyset$ and $\langle \mathcal{B} \rangle \setminus B \neq \emptyset$. Moreover, if $\langle \mathcal{A} \rangle \setminus A = \{a\}$ or $\langle \mathcal{B} \rangle \setminus B = \{b\}$, respectively, then $\tau(a) > 1$ and $\tau(b) > 1$.
(2) If $A_1, A_2 \subset \langle \mathcal{A} \rangle$, $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = \langle \mathcal{A} \rangle$, $B \subset \langle \mathcal{B} \rangle$, $C \subset \langle \mathcal{C} \rangle$ such that $A_1 \sim_\tau B$, $A_2 \sim_\tau C$, then there exist nonempty sets B_1 and C_1 such that $B_1 \subset \langle \mathcal{B} \rangle \setminus B$, $C_1 \subset \langle \mathcal{C} \rangle \setminus C$ and $B_1 \sim_\tau C_1$.

Then $\mathcal{S}$ is said to be an *admissible system* of MV-algebras.

Now we define a relation $\sim$ on the union $\bigcup_{t \in T} \mathcal{A}_t$ of an admissible system $\mathcal{S}$ of MV-algebras.

Definition 5. Let $\{\mathcal{S} = \mathcal{A}_t : t \in T\}$ be an admissible system of MV-algebras. For every pair of MV-algebras $\mathcal{A}_t$ and $\mathcal{A}_s$ we choice a pair of sets A and B of atoms such that $A \subset \langle \mathcal{A}_t \rangle$, $B \subset \langle \mathcal{A}_s \rangle$, and $A \sim_\tau B$.

(1) We define $0_{\mathcal{A}_t} \sim 0_{\mathcal{A}_s}$ and $1_{\mathcal{A}_t} \sim 1_{\mathcal{A}_s}$ whenever $A = \emptyset$ and $B = \emptyset$.
(2) If $x, y \in \mathcal{A}_t$, then $x \sim y$ if and only if $x = y$.
(3) If $x \in \mathcal{A}_t$, $y \in \mathcal{A}_s$ and $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$, then $x \sim y$ whenever $x = \bigvee_{i=1}^n p_i a_i$ and $y = \bigvee_{i=1}^n p_i b_i$, where $p_i \in \{0, 1, 2, \ldots, \tau(a_i)\}$ for $i = 1, 2, \ldots, n$.
(4) If $x \sim y$ then $x^\perp \sim y^\perp$.

Lemma 6. *Let $\{\mathcal{S} = \mathcal{A}_t : t \in T\}$ be an admissible system of MV-algebras. Then $0_{\mathcal{A}_t} \sim 0_{\mathcal{A}_s}$ and $1_{\mathcal{A}_s} \sim 1_{\mathcal{A}_t}$ for arbitrary $s, t \in T$.*

Proof: Let $s, t \in T$, $A \subset \langle A_t \rangle$ and $B \subset \langle \mathcal{A}_s \rangle$ such that $A \sim_\tau B$ and $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$. If we put $p_i = 0$ for any $i = 1, 2, \ldots, n$, then $0_{\mathcal{A}_t} = \bigvee_{i=1}^n 0a_i$, $0_{\mathcal{A}_s} = \bigvee_{i=1}^n 0b_i$ and so $0_{\mathcal{A}_t} \sim 0_{\mathcal{A}_s}$. From (4) of Definition 5 we have immediately $1_{\mathcal{A}_t} \sim 1_{\mathcal{A}_s}$. □

Theorem 7. *Let $\mathcal{S} = \{\mathcal{A}_t : t \in T\}$ be an admissible system of MV-algebras. The relation $\sim$ is an equivalence relation on $\bigcup_{t \in T} \mathcal{A}_t$.*

Proof: The reflexivity and symmetry are obvious. To prove the transitivity we assume that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{S}$ are different MV-algebras and $x \in \mathcal{A}$, $y \in \mathcal{B}$, $z \in \mathcal{C}$ such that $x \sim y$ and $y \sim z$. This follows that there exist sets A, B, C such that $A \subset \langle \mathcal{A} \rangle$, $B \subset \langle \mathcal{B} \rangle$, $C \subset \langle \mathcal{C} \rangle$ and $A \sim_\tau B$, $B \sim_\tau C$. If $A = \emptyset$, then $B = \emptyset$ and also $C = \emptyset$. Then either $x = 0_\mathcal{A}$ or $x = 1_\mathcal{A}$. If $x = 0_\mathcal{A}$ then $y = 0_\mathcal{B}$ and as well as $z = 0_\mathcal{C}$, hence $x \sim z$. Similarly if $x = 1_\mathcal{A}$.

Now let $A = \{a_1, a_2, \ldots, a_n\} \neq \emptyset$. Then $B = \{b_1, b_2, \ldots, b_n\} \neq \emptyset$ and $C = \{c_1, c_2, \ldots, c_n\} \neq \emptyset$. There are two possibilities: either $x = \bigvee_{i=1}^n p_i a_i$ and then $y = \bigvee_{i=1}^n p_i b_i$, or $x = \bigwedge_{i=1}^n (p_i a_i)^\perp$ and then $y = \bigwedge_{i=1}^n (p_i b_i)^\perp$, where $p_i \in \{0, 1, 2, \ldots, \tau(a_i)\}$ for $i = 1, 2, \ldots, n$. In the first case necessarily $z = \bigvee_{i=1}^n p_i c_i$, which gives that $x \sim z$. Similarly it is in the second case. □

Theorem 8. *Let $\mathcal{A}$ be an atomic σ-complete MV-algebra and $A = \{a_1, a_2, \ldots, a_n\} \subset \langle \mathcal{A} \rangle$. Let u, v be elements from $\mathcal{A}$ such that $v = \bigvee_{i=1}^n q_i a_i$, where $0 \le q_i \le \tau(a_i)$, $i = 1, 2, \ldots, n$. Then $u \le v$ if and only if $u = \bigvee_{i=1}^n p_i a_i$, where $0 \le p_i \le q_i$ for $i = 1, 2, \ldots, n$.*

Proof: Let $u \leq v$. First observe that there exists no atom $a \in \langle \mathcal{A} \rangle \setminus A$ such that $a \leq u$. Indeed, if an atom a like this existed, then

$$a = a \wedge u \leq a \wedge v = a \wedge \left(\bigvee_{i=1}^{n} q_i a_i \right) = \bigvee_{i=1}^{n} a \wedge (q_i a_i) = 0_{\mathcal{A}}.$$

Because $u \wedge (\tau(a_i)a_i) \leq \tau(a_i)a_i$, there exists p_i such that $0 \leq p_i \leq \tau(a_i)$, and hence $u \wedge (\tau(a_i)a_i) = p_i a_i$.

We put $\alpha = \bigvee_{i=1}^{n} \tau(a_i)a_i$. Then $u \leq v \leq \alpha$ and consequently,

$$u = u \wedge \alpha = u \wedge \left(\bigvee_{i=1}^{n} \tau(a_i)a_i \right) = \bigvee_{i=1}^{n} u \wedge (\tau(a_i)a_i) = \bigvee_{i=1}^{n} p_i a_i.$$

Suppose that there exists p_k such that $p_k > q_k$ for some $k \in \{1, 2, \ldots, n\}$. Then $p_k a_k > q_k a_k$. On the other hand we have

$$p_k a_k = (p_k a_k) \wedge u \leq (p_k a_k) \wedge v = (p_k a_k) \wedge \left(\bigvee_{i=1}^{n} q_i a_i \right)$$

$$= \bigvee_{i=1}^{n} ((p_k a_k) \wedge (q_i a_i)) = (p_k a_k) \wedge (q_k a_k) = q_k a_k,$$

which is contradiction with the assumption $p_k a_k > q_k a_k$, therefore, $p_i \leq q_i$ for all $i = 1, 2, \ldots, n$. □

Corollary 9. *Let $u = \bigwedge_{i=1}^{n} (p_i a_i)^{\perp}$, $0 \leq p_i \leq \tau(a_i)$. Then $u \leq v$ if and only if $v = \bigwedge_{i=1}^{n} (q_i a_i)^{\perp}$, where $0 \leq q_i \leq p_i$ for $i = 1, 2, \ldots, n$.*

Theorem 10. *Let $S = \{\mathcal{A}_t : t \in T\}$ be an admissible system of MV-algebras and $A \subset \langle \mathcal{A}_t \rangle$ and $B \subset \langle \mathcal{A}_s \rangle$ such that $A \sim_\tau B$. Let $x, u, \in \mathcal{A}_t$ and $y, v \in \mathcal{A}_s$ such that $x \sim y$ and $u \sim v$. The following assertions are true.*

(1) *$u \leq_t x$ if and only if $v \leq_s y$, and moreover, $x \ominus_t u \sim y \ominus_s v$, where $\leq_t$, $\leq_s$ are pratial orderings, and $\ominus_t$, $\ominus_s$, are differences on $\mathcal{A}_t$ and $\mathcal{A}_s$, respectively.*

(2) *$u \vee_t x \sim v \vee_s y$ and $u \wedge_t x \sim v \wedge_s y$, where $\vee_t (\vee_s)$ is the union and $\wedge_t (\wedge_s)$ is the meet on $\mathcal{A}_t (\mathcal{A}_s)$.*

Proof: For simplicity of notation, we shall write $\leq$ $(\ominus, \vee, \wedge)$ instead of $\leq_t$ and $\leq_s$ ($\ominus_t$ and $\ominus_s$, $\vee_t$ and $\vee_s$, $\wedge_t$ and $\wedge_s$).

(1) Let $\langle \mathcal{A}_t \rangle = \{a_1, a_2, \ldots, a_m\}$ be a set of all atoms of $\mathcal{A}_t$ and $A = \{a_1, a_2, \ldots, a_n\} \subset \langle \mathcal{A}_t \rangle$, where $n < m$, and $B = \{b_1, b_2, \ldots, b_n\} \subset \langle \mathcal{A}_s \rangle$. Put $\alpha = \bigvee_{i=1}^{n} \tau(a_i)a_i$ and $\beta = \bigvee_{i=1}^{n} \tau(b_i)b_i$. It is visible that $\alpha \sim \beta$ and $\alpha \vee \alpha^{\perp} = \bigvee_{i=1}^{m} \tau(a_i)a_i = 1_{\mathcal{A}_t}$.

There are only two possibilities: either $x = \bigvee_{i=1}^{n} p_i a_i$ or $x = \bigwedge_{i=1}^{n} (p_i a_i)^{\perp}$. If $x = \bigvee_{i=1}^{n} p_i a_i$, then $y = \bigvee_{i=1}^{n} p_i b_i$. The inequality $u \leq x$ implies that $u = \bigvee_{i=1}^{n} q_i a_i$, where $q_i \leq p_i$ for $i = 1, 2, \dots, n$. Then $v = \bigvee_{i=1}^{n} q_i b_i \leq \bigvee_{i=1}^{n} p_i b_i = y$.

Let us calculate

$$
\begin{aligned}
x \ominus u = x - u &= \left(\bigvee_{i=1}^{n} p_i a_i \right) - \left(\bigvee_{j=1}^{n} q_j a_j \right) \\
&= \bigvee_{i=1}^{n} \left(p_i a_i - \bigvee_{j=1}^{n} q_j a_j \right) = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{n} (p_i a_i - q_j a_j) \\
&= \bigvee_{i=1}^{n} (p_i a_i - q_1 a_1) \wedge \cdots \wedge (p_i a_i - q_i a_i) \wedge \cdots \wedge (p_i a_i - q_n a_n) \\
&= \bigvee_{i=1}^{n} (p_i a_i) \wedge \cdots \wedge ((p_i - q_i) a_i) \wedge \cdots \wedge (p_i a_i) = \bigvee_{i=1}^{n} (p_i - q_i) a_i .
\end{aligned}
$$

Likewise $y \ominus v = \bigvee_{i=1}^{n} (p_i - q_i) b_i$, consequently, $x \ominus u \sim y \ominus v$.

We note that if $x = \bigvee_{i=1}^{n} p_i a_i$ and $u = \bigwedge_{i=1}^{n} (q_i a_i)^{\perp}$, then $\alpha^{\perp} \leq u \leq x \leq \alpha$, which gives $\alpha^{\perp} \leq \alpha \wedge \alpha^{\perp} = 0_{\mathcal{A}_t}$, a contradiction.

Let $x = \bigwedge_{i=1}^{n} (p_i a_i)^{\perp}$. Then $y = \bigwedge_{i=1}^{n} (p_i b_i)^{\perp}$ and there are two possibilities: either $u = \bigwedge_{i=1}^{n} (q_i a_i)^{\perp}$ or $u = \bigvee_{i=1}^{n} q_i a_i$. In the first case in the inquality $u \leq x$ gives that $p_i \leq q_i$ for $i = 1, 2, \dots, n$. Then $v = \bigwedge_{i=1}^{n} (q_i b_i)^{\perp} \leq \bigwedge_{i=1}^{n} (p_i b_i)^{\perp} = y$ and

$$
x \ominus u = u^{\perp} \ominus x^{\perp} = \left(\bigvee_{j=1}^{n} q_j a_j \right) \ominus \left(\bigvee_{i=1}^{n} p_i a_i \right) = \bigvee_{j=1}^{n} (q_j - p_j) a_j .
$$

In like manner we obtain $y \ominus v = \bigvee_{j=1}^{n} (q_j - p_j) b_j$, which yields $x \ominus u \sim y \ominus v$.

Now we assume that $x = \bigwedge_{i=1}^{n} (p_i a_i)^{\perp}$ and $u = \bigvee_{i=1}^{n} q_i a_i$. Then

$$
\begin{aligned}
\bigvee_{i=1}^{n} q_i a_i = u = u \wedge \alpha \leq x \wedge \alpha &= x \wedge \left(\bigvee_{i=1}^{n} \tau(a_i) a_i \right) = \bigvee_{i=1}^{n} x \wedge \tau(a_i) a_i \\
&= \bigvee_{i=1}^{n} \left(\bigvee_{j=1}^{n} (p_j a_j)^{\perp} \right) \wedge \tau(a_i) a_i = \bigvee_{i=1}^{n} (p_i a_i)^{\perp} \wedge \tau(a_i) a_i \\
&= \bigvee_{i=1}^{n} (\tau(a_i) a_i \ominus p_i a_i) = \bigvee_{i=1}^{n} (\tau(a_i) \ominus p_i) a_i .
\end{aligned}
$$

Hence $q_i \leq \tau(a_i) \ominus p_i$, which gives $p_i + q_i \leq \tau(a_i)$ for $i = 1, 2, \dots, n$. The inequality $u \leq x$ implies that $x^\perp \oplus u$ exists in $\mathcal{A}_t$ and

$$x^\perp \oplus u = \left(\bigvee_{i=1}^{n} p_i a_i\right) \oplus \bigvee_{i=1}^{n} q_i a_i = \left(\bigoplus_{i=1}^{n} p_i a_i\right) \oplus \left(\bigoplus_{i=1}^{n} q_i a_i\right)$$
$$= \bigoplus_{i=1}^{n} (p_i + q_i) a_i = \left(\bigvee_{i=1}^{n} (p_i + q_i\right) a_i.$$

Then $x \ominus u = (x^\perp \oplus u)^\perp = \bigwedge_{i=1}^{n} ((p_i + q_i) a_i)^\perp$.

On the other hand we have $y \geq y \wedge \beta = \bigwedge_{i=1}^{n} (\tau(b_i) \ominus p_i) b_i \geq \bigvee_{i=1}^{n} q_i b_i = v$ and also $y \ominus v = \bigvee_{i=1}^{n} ((p_i + q_i) b_i)^\perp$, consequently (1) is proved.

(2) There are four possibilities.

(i) Let $x = \bigvee_{i=1}^{n} p_i a_i$ and $u = \bigvee_{i=1}^{n} q_i a_i$. Then

$$x \vee u = \left(\bigvee_{i=1}^{n} p_i a_i\right) \vee \left(\bigvee_{i=1}^{n} q_i a_i\right) = \bigvee_{i=1}^{n} (p_i a_i \vee q_i a_i)$$
$$= \bigvee_{i=1}^{n} (\max\{p_i, q_i\}) a_i$$

and in the same manner we obtain $y \vee v = \bigvee_{i=1}^{n} (\max\{p_i, q_i\}) b_i$, which yields $x \vee u \sim y \vee v$.

(ii) We note that $(qa)^\perp \vee pa = (\min\{q, \tau(a) - p\} a)^\perp$ for every atom a from $\mathcal{A}_t$, $0 \leq p \leq \tau(a)$, $0 \leq q \leq \tau(a)$. Indeed,

$$\begin{aligned}
(qa)^\perp \vee pa &= (\tau(a)a \ominus (\tau(a) - q)a)^\perp \vee pa \\
&= ((\tau(a)a)^\perp \oplus (\tau(a) - q)a) \vee pa \\
&= ((\tau(a)a)^\perp \vee (\tau(a) - q)a) \vee pa \\
&= (\tau(a)a)^\perp \vee ((\tau(a) - q)a \vee pa) \\
&= (\tau(a)a)^\perp \vee \max\{\tau(a) - q,\, p\} a \\
&= \tau(a)a^\perp \oplus \max\{\tau(a) - q,\, p\} a \\
&= (\tau(a)a \ominus \max\{\tau(a) - q,\, p\} a)^\perp \\
&= (\min\{q, \tau(a) - p\} a)^\perp.
\end{aligned}$$

If $x = \bigvee_{i=1}^{n} p_i a_i$ and $u = \bigwedge_{i=1}^{n} (q_i a_i)^{\perp}$, then

$$u \vee x = \bigwedge_{i=1}^{n} (q_i a_i)^{\perp} \vee \left(\bigvee_{j=1}^{n} p_j a_j \right) = \bigwedge_{i=1}^{n} \left((q_i a_i)^{\perp} \vee \left(\bigvee_{j=1}^{n} p_j a_j \right) \right)$$

$$= \bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{n} p_j a_j \vee (q_i a_i)^{\perp} \right) = \bigwedge_{i=1}^{n} (p_i a_i \vee (q_i a_i)^{\perp})$$

$$= \bigwedge_{i=1}^{n} (\min\{q_i, \tau(a_i) - p_i\} a_i)^{\perp}.$$

Similarly $v \vee y = \bigwedge_{i=1}^{n} (\min\{q_i, \tau(b_i) - p_i\} b_i)^{\perp}$, so that $u \vee x \sim v \vee y$.

(iii) If $x = \bigwedge_{i=1}^{n} (p_i a_i)^{\perp}$ and $u = \bigvee_{i=1}^{n} q_i a_i$, then $x \vee u = \bigwedge_{i=1}^{n} (\min\{p_i, \tau(a_i) - q_i\} a_i)^{\perp}$ and $y \vee v = \bigwedge_{i=1}^{n} (\min\{p_i, \tau(b_i) - q_i\} b_i)^{\perp}$.

(iv) Finally, let $x = \bigwedge_{i=1}^{n} (p_i a_i)^{\perp}$ and $u = \bigwedge_{i=1}^{n} (q_i a_i)^{\perp}$. Then

$$x \vee u = (x^{\perp} \wedge u^{\perp})^{\perp} = \left(\left(\bigvee_{i=1}^{n} p_i a_i \right) \wedge \left(\bigvee_{j=1}^{n} q_j a_j \right) \right)^{\perp}$$

$$= \left(\bigvee_{i=1}^{n} (p_i a_i \wedge q_i a_i) \right)^{\perp} = \left(\bigvee_{i=1}^{n} \min\{p_i, q_i\} a_i \right)^{\perp}$$

$$= \bigwedge_{i=1}^{n} (\min\{p_i, q_i\} a_i)^{\perp},$$

and by analogy $y \vee v = \bigwedge_{i=1}^{n} (\min\{p_i, q_i\} b_i)^{\perp}$. With respect to (4) of Definition 5, the proof of complete. □

Let $\overline{x}$ be the equivalence class determined by x and $\mathcal{P}$ be the quotient set, i.e.

$$\overline{x} = \left\{ y \in \bigcup_{t \in T} \mathcal{A}_t : y \sim x \right\} \quad \text{and} \quad \mathcal{P} = \left\{ \overline{x} : x \in \bigcup_{t \in T} \mathcal{A}_t \right\}.$$

The set $\mathcal{P}$ is called an *MV-algebras pasting*.

If we denote $\overline{\mathcal{A}_t} = \{\overline{x} : x \in \mathcal{A}_t\}$, then $\mathcal{P} = \bigcup_{t \in T} \overline{\mathcal{A}_t}$. We prove that an MV-algebras pasting $\mathcal{P}$ is a D-poset. To prove this, we first define a partial ordering on $\mathcal{P}$.

Definition 11. Let $\mathcal{P}$ be an MV-algebras pasting and $\overline{x}, \overline{y} \in \mathcal{P}$. Then $\overline{x} \leq \overline{y}$ if and only if there exist an MV-algebra $\mathcal{A}_r$ and elements $u, v \in \mathcal{A}_r$ such that $u \in \overline{x}$, $v \in \overline{y}$ and $u \leq_r v$.

Theorem 12. *The relation $\leq$ is a partial ordering on $\mathcal{P}$.*

Proof: The reflexivity is visible.

Let $x \in \mathcal{A}_t$ and $y \in \mathcal{A}_s$ such that $\overline{x} \leq \overline{y}$ and $\overline{y} \leq \overline{x}$. Then there are MV-algebras $\mathcal{A}_{r_1}$, $\mathcal{A}_{r_2}$ for $r_1, r_2 \in T$, and elements u_1, v_1, u_2, v_2 such that $u_1, v_1 \in \mathcal{A}_{r_1}$ and $x \sim u_1, v_1 \sim y, u_1 \leq_{r_1} v_1$ and $u_2, v_2 \in \mathcal{A}_{r_2}$ and $y \sim u_2, v_2 \sim x, u_2 \leq_{r_2} v_2$. On the other hand we have that $u_1 \sim v_2, v_1 \sim u_2$ and, considering Theorem 10, we obtain that $v_2 \leq_{r_2} u_2$, which gives $v_2 = u_2$ and hence $\overline{x} = \overline{v}_2 = \overline{u}_2 = \overline{y}$.

Now we prove the transitivity. Suppose that $x \in \mathcal{A}_t$, $y \in \mathcal{A}_s$ and $z \in \mathcal{A}_r$ such that $\overline{x} \leq \overline{y}$ and $\overline{y} \leq \overline{z}$. Then there exist MV-algebras $\mathcal{A}_{r_1}$ and $\mathcal{A}_{r_2}$ and elements u_1, $v_1 \in \mathcal{A}_{r_1}$ and $u_2, v_2 \in \mathcal{A}_{r_2}$ such that $x \sim u_1, y \sim v_1, y \sim v_2, z \sim u_2$ and $u_1 \leq_{r_1} v_1$ and $v_2 \leq_{r_2} u_2$. Because of $v_1 \sim v_2$, there exist sets A and B such that $A \subset \langle \mathcal{A}_{r_1} \rangle$, $B \subset \langle \mathcal{A}_{r_2} \rangle$ and $A \sim_\tau B$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. There are two possibilities: either $v_1 = \bigvee_{i=1}^n q_i a_i$ or $v_1 = \bigwedge_{i=1}^n (q_i a_i)^\perp$, where $0 \leq q_i \leq \tau(a_i)$ for $i = 1, 2, \ldots, n$.

If $v_1 = \bigvee_{i=1}^n q_i a_i$, then $v_2 = \bigvee_{i=1}^n q_i b_i$. The inequality $u_1 \leq_{r_1} v_1$ implies that $u_1 = \bigvee_{i=1}^n p_i a_i$, where $0 \leq p_i \leq q_i$ for $i = 1, 2, \ldots, n$. We put $u = \bigvee_{i=1}^n p_i b_i$. Then $u \in \mathcal{A}_{r_2}$ and $u \sim u_1 \sim x$, $u \leq_{r_2} v_2$ and $u_2 \sim z$, which gives that $\overline{x} \leq \overline{z}$.

If $v_1 = \bigwedge_{i=1}^n (q_i a_i)^\perp$ then $v_2 = \bigwedge_{i=1}^n (q_i b_i)^\perp$. Using Corollary 9 we obtain $u_2 = \bigwedge_{i=1}^n (p_i b_i)^\perp$, where $0 \leq p_i \leq q_i$ for $i = 1, 2, \ldots, n$. Putting $u = \bigwedge_{i=1}^n (p_i a_i)^\perp$ we get $u \in \mathcal{A}_{r_1}$, $u \sim u_2 \sim z$ and $u_1 \leq_{r_1} v_1 \leq_{r_1} u$, accordingly, $\overline{x} \leq \overline{z}$. □

Corollary 13. *Let $\overline{x}, \overline{y}, \overline{z} \in \mathcal{P}$ such that $\overline{x} \leq \overline{y} \leq \overline{z}$. Then there exist an MV-algebra $\mathcal{A}_t$ and elements $u, v, w \in \mathcal{A}_t$ such that $u \in \overline{x}$, $v \in \overline{y}$, $w \in \overline{z}$ and $u \leq_t v \leq_t w$.*

It is evident that the MV-algebras pasting $\mathcal{P}$ is a partially ordered set with the greatest element $\overline{1_{\mathcal{A}_t}}$ (we shall denote it by $1_\mathcal{P}$) and the least element $\overline{0_{\mathcal{A}_i}}$ (we shall denote it by $0_\mathcal{P}$).

Now we shall define a partial binary operation $\ominus$ on $\mathcal{P}$ as follows. Let $\overline{x}$, $\overline{y} \in \mathcal{P}$ such that $\overline{x} \leq \overline{y}$. Then there exist an MV-algebra $\mathcal{A}_t$ and elements $u, v \in \mathcal{A}_t$ such that $u \in \overline{x}$, $v \in \overline{y}$ and $u \leq_t v$. We put

$$\overline{y} \ominus \overline{x} = \overline{v \ominus_t u},$$

where $\ominus_t$ is a partial difference operation on the MV-algebra $\mathcal{A}_t$. It is easy to verify that $\ominus$ satisfies the axioms (D1) and (D2) of the difference operation. Now it is visible that the following theorem is true.

Theorem 14. *Let $\mathcal{P}$ be an MV-algebras pasting of an admissible system $\mathcal{S}$. Then $(\mathcal{P}, \leq, 1_\mathcal{P}, 0_\mathcal{P}, \ominus)$ is a D-poset.*

Theorem 15. *Let $\{\mathcal{A}, \mathcal{B}\}$ be an admissible system of MV-algebras. Then the MV-algebras pasting $\mathcal{P} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$ is a D-lattice.*

Proof: It is not difficult to prove that $\overline{\mathcal{A}}$, $\overline{\mathcal{B}}$ and $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ are sub-MV-algebras of $\mathcal{P}$.

Let $A \subset \langle \mathcal{A} \rangle$ and $B \subset \langle \mathcal{B} \rangle$ such that $A \sim_\tau B$.

(i) At first we assume that $A = \emptyset$ and $B = \emptyset$. Then $\overline{\mathcal{A}} \cap \overline{\mathcal{B}} = \{0_\mathcal{P}, 1_\mathcal{P}\}$. If $\overline{x}, \overline{y} \in \mathcal{P}$ such that $\overline{x}, \overline{y} \in \overline{\mathcal{A}}$ (or $\overline{x}, \overline{y} \in \overline{\mathcal{B}}$) then $\overline{x} \vee \overline{y} \in \overline{\mathcal{A}} \subset \mathcal{P}$ (or $\overline{x} \vee \overline{y} \in \overline{\mathcal{B}} \subset \mathcal{P}$).

Let $\overline{x} \in \overline{\mathcal{A}} \setminus \overline{\mathcal{B}}$ and $\overline{y} \in \overline{\mathcal{B}} \setminus \overline{\mathcal{A}}$. We prove that $\overline{x} \vee \overline{y} = 1_\mathcal{P}$. If $\overline{w} \in \mathcal{P}$ such that $\overline{x} \leq \overline{w}$ and $\overline{y} \leq \overline{w}$ then the first inequality gives $\overline{w} \in \overline{\mathcal{A}}$ and the second one $\overline{w} \in \overline{\mathcal{B}}$, therefore $\overline{w} \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Hence $\overline{w} = 1_\mathcal{P}$.

(ii) Let $\langle \mathcal{A} \rangle = \{a_1, a_2, \ldots, a_m\}$ be a set of all atoms of $\mathcal{A}$ and $A = \{a_1, a_2, \ldots, a_n\} \subset \langle \mathcal{A} \rangle$, where $n < m$, and $B = \{b_1, b_2, \ldots, b_n\} \subset \langle \mathcal{B} \rangle = \{b_1, b_2, \ldots, b_k\}$, where $n < k$.

We put $\alpha_1 = \bigvee_{i=1}^n \tau(a_i)a_i$ and $\beta_1 = \bigvee_{i=1}^n \tau(b_i)b_i$. Let us denote by $\overline{\alpha}$ and $\overline{x_i}$ the equivalence classes such that $\alpha_1, \beta_1 \in \overline{\alpha}$ and $a_i, b_i \in \overline{x_i}$, for $i = 1, 2, \ldots, n$. Evidently, $\overline{x_i}$ are atoms in $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ for $i = 1, 2, \ldots, n$. We prove that $\langle \overline{\mathcal{A}} \cap \overline{\mathcal{B}} \rangle = \{\overline{x_1}, \overline{x_2}, \ldots \overline{x_n}, \overline{\alpha}^\perp\}$ is the set of all atoms of $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Indeed, if there exists $\overline{x_k} \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ for some $k \in \{1, 2, \ldots, n\}$ such that $\overline{x_k} \leq \overline{\alpha}^\perp$, then $a_k \leq_\mathcal{A} \alpha_1^\perp = \bigwedge_{i=1}^n (\tau(a_i)a_i)^\perp \leq_\mathcal{A} (\tau(a_k)a_k)^\perp$, which contradicts the maximality of $(\tau(a_k)a_k)$.

Suppose that $\overline{x} \in \overline{\mathcal{A}} \setminus \overline{\mathcal{B}}$, $\overline{y} \in \overline{\mathcal{B}} \setminus \overline{\mathcal{A}}$, $u \in \mathcal{A}$ and $v \in \mathcal{B}$ such that $u \in \overline{x}$ and $v \in \overline{y}$. Then

$$u = \bigvee_{i=1}^m p_i a_i = \bigvee_{i=1}^n p_i a_i \vee \bigvee_{i=n+1}^m p_i a_i = u_1 \vee u_2,$$

where $\overline{u_1} \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ and $\overline{u_2} \in \overline{\mathcal{A}} \setminus \overline{\mathcal{B}}$. Likewise

$$v = \bigvee_{i=1}^k q_i b_i = \bigvee_{i=1}^n q_i b_i \vee \bigvee_{i=n+1}^k q_i b_i = v_1 \vee v_2,$$

where $\overline{v_1} \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ and $\overline{v_2} \in \overline{\mathcal{B}} \setminus \overline{\mathcal{A}}$. Visibly $\overline{u_1} \vee \overline{v_1} \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$.

Set $\overline{z} = \overline{u_1} \vee \overline{v_1}$, $\alpha_2 = \bigvee_{i=n+1}^m \tau(a_i)a_i$ and $\beta_2 = \bigvee_{i=n+1}^k \tau(b_i)b_i$. Because

$$1_\mathcal{A} = \bigvee_{i=1}^m \tau(a_i)a_i = \alpha_1 \vee \alpha_2 = \alpha_1 \oplus \alpha_2,$$

we have $\alpha_2 = \alpha_1^\perp$ and also $\beta_2 = \beta_1^\perp$. Further $u_2 \leq \alpha_2 = \alpha_1^\perp$ and $v_2 \leq \beta_2 = \beta_1^\perp$, thence it follows $\overline{u_2} \leq \overline{\alpha}^\perp$ and $\overline{v_2} \leq \overline{\alpha}^\perp$. Then $\overline{z} \vee \overline{\alpha}^\perp \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ and $\overline{x} \leq \overline{z} \vee \overline{\alpha}^\perp$ as well as $\overline{y} \leq \overline{z} \vee \overline{\alpha}^\perp$.

Let $\overline{w} \in \mathcal{P}$ such that $\overline{x} \leq \overline{w}$ and $\overline{y} \leq \overline{w}$. Then $\overline{w} \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ and there are $w_1 \in \mathcal{A}$ and $w_2 \in \mathcal{B}$ such that $w_1, w_2 \in \overline{w}$. The inequality $\overline{x} \leq \overline{w}$ implies that $u \leq_\mathcal{A} w_1$

and hence $u_1 \leq_{\mathcal{A}} w_1$. Analogically $v_1 \leq_{\mathcal{B}} w_2$, which gives $\overline{u_1} \leq \overline{w}$ and $\overline{v_1} \leq \overline{w}$, so $\overline{z} = \overline{u_1} \vee \overline{v_1} \leq \overline{w}$.

Now we prove that $\overline{\alpha}^{\perp} \leq \overline{w}$. Because

$$\overline{w} \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}} \quad \text{and} \quad \langle \overline{\mathcal{A}} \cap \overline{\mathcal{B}} \rangle = \left\{ \overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}, \overline{\alpha}^{\perp} \right\}$$

we get

$$\overline{w} = \left(\bigvee_{i=1}^{n} r_i \overline{x_1} \right) \vee r_0 \overline{\alpha}^{\perp},$$

where $0 \leq r_i \leq \tau(\overline{x_1}) = \tau(a_i) = \tau(b_i)$ and $r_0 \in \{0, 1\}$. If $r_0 = 0$, then using $\overline{u_2} \leq \overline{w}$ we obtain

$$\overline{u_2} = \overline{u_2} \wedge \overline{\alpha}^{\perp} \leq \overline{w} \wedge \overline{\alpha}^{\perp} = \left(\bigvee_{i=1}^{n} r_i \overline{x_1} \right) \wedge \overline{\alpha}^{\perp}$$

$$= \bigvee_{i=1}^{n} \left(r_i \overline{x_1} \wedge \overline{\alpha}^{\perp} \right) = 0_{\mathcal{P}},$$

a controversy. Then $\overline{w} \geq \overline{\alpha}^{\perp}$, thus $\overline{w} \geq \overline{z} \vee \overline{\alpha}^{\perp}$. We proved that $\overline{x} \vee \overline{y} = \overline{z} \vee \overline{\alpha}^{\perp}$ and so $\mathcal{P}$ is a D-lattice. □

The properties of an MV-algebras pasting depend on the choice of the equivalent sets (with respect to isotropic indices) and on types of pasting MV-algebras.

Let $\mathcal{S} = \{\mathcal{A}_t : t \in T\}$ be an admissible system of MV-algebras and $A_t \subset \langle \mathcal{A}_t \rangle$, $A_s \subset \langle \mathcal{A}_s \rangle$ such that $A_t = \emptyset$ and $A_s = \emptyset$ for every $s, t \in T$. Then the MV-algebras pasting $\mathcal{P}$ of the system $\mathcal{S}$ is called the 0-1-pasting. Every MV-algebras 0-1-pasting is a D-lattice, especially, a Boolean algebras 0-1-pasting is an orthomodular lattice.

Theorem 16. *Let $\mathcal{S} = \{\mathcal{A}_t : t \in T\}$ be an admissible system of MV-algebras and $\mathcal{P} = \bigcup_{t \in T} \overline{\mathcal{A}_t}$ be an MV-algebras pasting. If $\mathcal{P}$ is a lattice, then $\overline{\mathcal{A}_t}$ are blocks in $\mathcal{P}$ for all $t \in T$.*

Proof: We prove that every $\overline{\mathcal{A}_t}$ is the maximal compatible set in $\mathcal{P}$.

Let $\overline{z} \in \mathcal{P}$ and $\overline{z} \leftrightarrow \overline{x}$ for all $\overline{x} \in \overline{\mathcal{A}_t}$. Let us assume that $z \notin \overline{\mathcal{A}_t}$. Then there exists an MV-algebra $\overline{\mathcal{A}_s} \in \mathcal{P}$ such that $\overline{z} \in \overline{\mathcal{A}_s} \setminus \overline{\mathcal{A}_t}$. We choose $\overline{x}$ such that $\overline{x} \in \overline{\mathcal{A}_t} \setminus \overline{\mathcal{A}_s}$. Let $A \subset \langle \mathcal{A}_t \rangle$ and $B \subset \langle \mathcal{A}_s \rangle$ such that $A \sim_{\tau} B$.

(i) If $A = \emptyset$ and $B = \emptyset$ then $\overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_s} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}\}$. It follows that $\overline{z} \vee \overline{x} = 1_{\mathcal{P}}$ and $\overline{z} \wedge \overline{x} = 0_{\mathcal{P}}$. The compatibility of $\overline{z}$ and $\overline{x}$ gives $(\overline{z} \vee \overline{x}) \ominus \overline{z} = \overline{x} \ominus (\overline{z} \wedge \overline{x})$, that is $1_{\mathcal{P}} \ominus \overline{z} = \overline{x} \in \overline{\mathcal{A}_t}$. This contradicts our assumption that $\overline{z} \notin \overline{\mathcal{A}_t}$.

(ii) Let $A \neq \emptyset$ and $a_0 \in \langle \mathcal{A}_t \rangle \setminus A$. Then $\overline{a_0} \in \overline{\mathcal{A}_t} \setminus \overline{\mathcal{A}_s}$ and by the assumption we have $\overline{z} \leftrightarrow \overline{a_0}$. Because $\overline{z} \wedge \overline{a_0} = 0_{\mathcal{P}}$ and $\overline{z} \vee \overline{a_0} \in \overline{\mathcal{A}_t} \cap \overline{\mathcal{A}_s}$, we get $\overline{z} = \overline{z} \ominus \overline{z} \wedge \overline{a_0} = (\overline{z} \vee \overline{a_0}) \ominus \overline{a_0} \in \overline{\mathcal{A}_t}$, a contradiction.

We proved that the compatibility $\overline{z} \leftrightarrow \overline{x}$ for all $\overline{x} \in \overline{\mathcal{A}_t}$ follows $\overline{z} \in \overline{\mathcal{A}_t}$, which vindicates the maximality of $\overline{\mathcal{A}_t}$. □

There remains the open problem to establish the necessary and sufficient conditions such that an MV-algebras pasting is a lattice.

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REFERENCES

Chang, C. C. (1957). Algebraic analysis of many valued logics. *Transactions of the American Mathematical Society*, **88**, 467–490.

Chovanec, F. and Kôpka, F. (1997). Boolean D-posets. *Tatra Mountains Mathematical Publications* **10**, 183–197.

Dvurečenskij, A. and Pulmannová, S. (2000). *New trends in Quantum Structures*, Kluwer, Dordrecht, The Netherlands.

Font, J. M., Rodrígues, A. J., and Torrens, A. (1984). *Wajsberg algebras. Stochastica* **8**, 5–31.

Foulis, D. J. and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics. *Foundations of Physics* **24**, 1331–1352.

Giuntini, R. and Greuling, H. (1989). Toward a formal language for unsharp properties. *Foundations of Physics* **20**, 931–945.

Greechie, R. (1971). Orthomodular lattices admitting no states. *Journal of Combinatorial Theory* **10**, 119–132.

Jenča, G. (2001). Blocks of homogeneous effect algebras. *Bulletin of the Australian Mathematical Society* **64**, 81–98.

Kôpka, F. (1992). D-posets of fuzzy sets. *Tatra Mountains Mathematical Publications* **1**, 83–88.

Kôpka, F. (1995). *Compatibility in D-posets. International Journal of Theoretical Physics* **34**, 1525–1531.

Riečanová, Z. (2000). Generalization of blocks for D-lattice and lattice ordered effect algebras. *International Journal of Theoretical Physics* **39**, 231–237.